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Moment Systems and Orthogonal Polynomials in Several Variables

PHILIP FEINSILVER

Department of Mathematics, Southern Illinois University, Carbondale, Illinois 62901

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The first part of this paper deals with general moment ("Appell") systems on $\mathbb{R}^N$ generated by a Hamiltonian function $H(x, D)$ and also with representations of $GL(N)$ on the associated spaces of polynomials. The second part discusses the theory of Bernoulli generators on $\mathbb{R}^N$ determining systems of orthogonal polynomials that are extensions of the Meixner polynomials to several variables. Linear actions for these spaces are discussed. Some tensors related to the general Bernoulli generators are considered.

I. INTRODUCTION: CALCULUS OF SEQUENCES

Our presentation is based on the following observations. Let c_k denote an arbitrary sequence and consider the (formal) generating function $e(z) = \sum_{k=0}^{\infty} (z^k/k!)\, c_k$. Then the operators multiplication by z and d/dz act as follows:

$$ze(z)\colon c_k \to kc_{k-1}, \tag{1}$$

$$\frac{d}{dz} e(z)\colon c_k \to c_{k+1}. \tag{2}$$

We can formalize this by introducing an operator C such that $[z, C] = zC - Cz = 1$. Then we may set $c_k = C^k c_0$ and $e(z) = e^{zC} c_0$. In this exponential form it is apparent that $z \leftrightarrow d/dC$, $d/dz \leftrightarrow C$. The crucial observation is that there is an important class of sequences $c_k(x)$ for which the operator C may be explicitly determined as a function of x and d/dx.

The basic construction is to start with a "Hamiltonian," a function $H(x, D)$, $D = d/dx$, which we assume to be analytic in its arguments with all the D's shifted to the right. (We refer the reader to [3] and [4] for basic operator calculus used in the sequel.) We set $x(t) = e^{tH} x e^{-tH}$ and $z(t) = e^{tH} D e^{-tH}$. Then the operator multiplication by z is represented by $z(t)$

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and the operator C by $x(t)$. If we specialize to $H = L(D) = L(z)$, a function analytic in a neighborhood of O in $\mathbb{C}$, $L(0) = 0$, then $h_k(x, t) \equiv c_k(x) = x(t)^k 1$ are a family of polynomials satisfying

$$\frac{\partial h_k}{\partial t} = L(D)\, h_k, \tag{1}$$

$$Dh_k = kh_{k-1}, \tag{2}$$

$$Ch_k = h_{k+1}, \tag{3}$$

$$CDh_k = kh_k. \tag{4}$$

L is called the *generator* and we call the $h_k(x, t)$ the *moment polynomials* associated with L.

The present study considers sequences $c_k(x_1, x_2, \ldots, x_N)$; i.e., $c_k(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^N$, to see how the above considerations extend for $N > 1$.

In Part II we will discuss vacuum functions and their associated moment systems and linear actions. Part III will deal with Bernoulli generators on $\mathbb{R}^N$ associated with systems of orthogonal polynomials that are isomorphic to moment systems.

II. Moment Systems in Several Variables

We have $\mathbf{x} \in \mathbb{R}^N$, $z_j = D_j = \partial/\partial x_j$, $[z_j, x_k] = \delta_{jk}$. Given a Hamiltonian $H = L(\mathbf{z})$, $C_j = x_j(t) = x_j + tL_j(\mathbf{z})$, $L_j = \partial L/\partial z_j$. $A = \mathbf{C} \cdot \mathbf{D} = \sum_{j=1}^N x_j(t)\, z_j$ is the *number operator.*

Vacuum Functions

Above, we chose $c_0 = 1$, i.e., the function which is indentically 1. By choosing various zero-level or *vacuum* functions we generate different sequences. We assume H is given.

Definition 1. A function u is *harmonic* if it satisfies $u_t + Hu = 0$; *coharmonic* if $u_t = Hu$.

2. A coharmonic function $\Omega(x, t)$ such that $\Omega(x, 0)$ is homogeneous of degree zero is a *vacuum function.*

3. A vacuum function independent of t is an *absolute vacuum.*

Remark. An absolute vacuum satisfies $\mathbf{x} \cdot \mathbf{D}\Omega = 0$, $H\Omega = 0$.

Vacuum functions are just those functions that have the following properties. Set $h_0 = \Omega$, a vacuum, and $h_\mathbf{n} = \mathbf{C}^\mathbf{n}\Omega$. Then

(1) $h_\mathbf{n}$ are coharmonic for all $\mathbf{n}$, $n_j \geqslant 0$, $1 \leqslant j \leqslant N$.

(2) $Ah_\mathbf{n} = (\sum_j n_j)\, h_\mathbf{n} = |\mathbf{n}|\, h_\mathbf{n}$.

(See Theorem 1 and Proposition 55 of [4].) For $N > 1$ there are two important extensions of the basic vacuums. In the following, by homogeneous we mean $f(\lambda \mathbf{x}) = \lambda^p f(\mathbf{x})$ for some p, for all real $\lambda > 0$.

PROPOSITION 1. *Let Ω be homogeneous of degree m. Set $\Omega_m = e^{tH}\Omega$ and $u_{\mathbf{n}}^{(m)} = \mathbf{C}^{\mathbf{n}}\Omega_m$. Then*

(1) *$u_{\mathbf{n}}^{(m)}$ are coharmonic.*

(2) *$A\, u_{\mathbf{n}}^{(m)} = (m + |\mathbf{n}|)\, u_{\mathbf{n}}^{(m)}$.*

Remark. Observe that Ω_m has the form $e^{tH} r^m \Omega_0$ where $r = |\mathbf{x}|$ and Ω_0 is homogeneous of degree zero. Recall that

$$A = \mathbf{C} \cdot \mathbf{D} = e^{tH} r \frac{d}{dr} e^{-tH}.$$

Proof. By the remark,

$$u_{\mathbf{n}}^{(m)} = e^{tH}\, \mathbf{x}^{\mathbf{n}}\, e^{-tH} e^{tH} r^m \Omega_0 = e^{tH} \mathbf{x}^{\mathbf{n}} r^m \Omega_0.$$

Thus we have,

$$A u_{\mathbf{n}}^{(m)} = e^{tH}\, r \frac{d}{dr}\, \mathbf{x}^{\mathbf{n}} r^m \Omega_0 = e^{tH}(|\mathbf{n}| + m + 0)\, \mathbf{x}^{\mathbf{n}} r^m \Omega_0.$$

We will call Ω_m a vacuum function of degree m. This simple extension immediately shows the significance of spherical functions, considering e.g., $H =$ Laplacian. By considering $H - \lambda$ instead of H we obtain the next extension.

PROPOSITION 2. *Let $\Omega_{m,\lambda}$ be a vacuum function of degree m for $H - \lambda$. Set $u_{\mathbf{n},\lambda}^{(m)} = \mathbf{C}^{\mathbf{n}}\Omega_{m,\lambda}$. Then*

(1) *$e^{\lambda t} u_{\mathbf{n},\lambda}^{(m)}$ is coharmonic for all $\mathbf{n}$.*

(2) *$A u_{\mathbf{n},\lambda}^{(m)} = (m + |\mathbf{n}|)\, u_{\mathbf{n},\lambda}^{(m)}$, where A is the number operator corresponding to H.*

This proposition follows as above. The eigenfunctions of H homogeneous of degree m are exactly the absolute vacuums $\Omega_{m,\lambda}$. The functions $u_{\mathbf{n},\lambda}^{(m)}$ are the powers $\mathbf{x}^{\mathbf{n}}$ evolved t units in time relative to a "ground state" $\Omega_{m,\lambda}$. These systems are essentially canonical bases for solutions of evolution equations of the form $u_t = Hu$.

Absolute vacuums $\Omega_{m,\lambda}$ are the solutions to systems of the form

$$H\Omega_{m,\lambda} = \lambda \Omega_{m,\lambda},$$

$$\mathbf{x} \cdot \mathbf{D}\Omega_{m,\lambda} = m\Omega_{m,\lambda}$$

Remark. One can construct vacuums of mixed homogeneity. Partition the variables $x_1, x_2, \ldots, x_N$ into classes $(\mathbf{x})_j$. Then set

$$\Omega_{\mathbf{m}} = \prod_j \Omega_{m_j}((\mathbf{x})_j),$$

where $\Omega_{m_j}((\mathbf{x})_j)$ is a vacuum of degree m_j in the jth group of variables. We would then have

$$A\, \mathbf{C}^{\mathbf{n}}\Omega_{\mathbf{m}} = (|\mathbf{m}| + |\mathbf{n}|)\, \mathbf{C}^{\mathbf{n}}\Omega_{\mathbf{m}}.$$

Thus for $N > 1$ we have a fascinating variety of moment systems. Some simple explicit results follow.

(1) $L = D_x + D_y$. We want to solve

(i) $f_x + f_y = \lambda f$,

(ii) $x f_x + y f_y = m f$.

Changing to polar coordinates and setting $f = r^m \Omega$, where Ω is homogeneous of degree zero it is easy to see that $\lambda = 0$. We then easily get directly from (i) and (ii) that, modulo a constant factor,

$$f = (y - x)^m.$$

Thus

$$u_{k,l}^{(m)} = e^{tL} x^k y^l (y-x)^m = (x+t)^k (y+t)^l (y-x)^m.$$

(2) $L = D_x^2 + D_y^2$. In polar coordinates we have

$$m^{(2)} r^{m-2}\Omega + m r^{m-2}\Omega + r^{m-2}\Omega'' = \lambda r^m \Omega.$$

Therefore $\lambda = 0$, and $\Omega'' + m^2\Omega = 0$. Thus a basis for vacuums of degree m is: $r^m \cos m\theta$, $r^m \sin m\theta$ for $m > 0$, and 1, θ for $m = 0$.

(3) $L = D_x^p + D_y^p$ for a positive integer p. We can factor

$$L = \prod_{j=0}^{p-1} (D_x - \alpha_j D_y), \qquad \text{where } \alpha_j = \exp[(2j+1)\pi i/p].$$

Solutions to $Lf = 0$ have as a basis arbitrary functions of the form $f_j(x, y) = f_j(x + \alpha_j y)$. Thus a basis for vacuums of degree $m > 0$ is provided by $\{\mathrm{Re}(x + \alpha_j y)^m, \mathrm{Im}(x + \alpha_j y)^m\}$, where we do not include repetitions arising from the relations $\bar{\alpha}_j = \alpha_{p-j-1}$. A basis for vacuums of degree zero is provided by

$$\left\{1, \mathrm{Re} \log \frac{x + \alpha_j y}{x + \alpha_k y}, \mathrm{Im} \log \frac{x + \alpha_j y}{x + \alpha_k y}\right\}_{j \neq k}.$$

Since $\alpha_j = \cos\theta_j + i\sin\theta_j$, $\theta_j = (2j+1)\pi/p$, we have: for $m > 0$: $\{\rho_j^m \cos m\psi_j(x,y),\ \rho_j^m \sin m\psi_j(x,y)\}_{0<j<p}$, where $\rho_j^2 = x^2 + y^2 + 2xy\cos\theta_j$, $\psi_j(x,y) = \arctan(y\sin\theta_j/(x + y\cos\theta_j))$, for $m = 0$: $\{1, \log(\rho_j/\rho_k),\ \psi_j - \psi_k\}$. For example, for $p = 3$ we have as bases: $m > 0$: $\{(x-y)^m$, $(x^2+xy+y^2)^{m/2}\sin[m\arctan\gamma]$, $(x^2+xy+y^2)^{m/2}\cos[m\arctan\gamma]\}$ with

$$\gamma = y\sqrt{3}/(2x+y),$$

$$m = 0:\ \left\{1, \log\frac{(x-y)^2}{x^2+xy+y^2}, \arctan\frac{y\sqrt{3}}{2x+y}\right\}.$$

Significance of Moment Systems: Local Solutions

If we want to solve $u_t = Hu$, we might look for a fundamental solution $p_t(\mathbf{x},\mathbf{y}) = e^{tH}\delta(\mathbf{x}-\mathbf{y})$. For $H = L(\mathbf{D})$ we would have

$$p_t(\mathbf{y}-\mathbf{x}) = e^{tL}\delta(\mathbf{x}-\mathbf{y}), \qquad \text{where} \quad p_t(\mathbf{x}) = 1/(2\pi)^N \int_{\mathbb{R}^N} e^{-i\boldsymbol{\xi}\cdot\mathbf{x}} e^{tL(i\boldsymbol{\xi})}\, d\boldsymbol{\xi}.$$

Then given a suitable function f we have

$$u = \int_{\mathbb{R}^N} f(\mathbf{y})\, p_t(\mathbf{x},\mathbf{y})\, dy \qquad \text{as the solution to } u_t = Hu,\ u_0 = f.$$

The solution thus is "global." That is, we are required to integrate over all of $\mathbb{R}^N$. The vacuum functions, however, allow the construction of solutions by applying the *local* operators C_j. The initial functions f are then restricted, e.g., to powers $\mathbf{x}^{\mathbf{n}}$ times Ω, and thus to analytic functions times the vacuum. The solutions generated are then *not* necessarily global; they will be well defined on the unit sphere. A ready example is to consider $L = \frac{1}{2}(D_x^2 + D_y^2)$. $\Omega = \arctan(y/x)$. Then $u_{01} = (y + t\partial_y)\,\Omega = y\arctan(y/x) + tx/(x^2+y^2)$ which satisfies the heat equation on any domain not containing $(0,0)$. Since the solution is constructed by applying the local operator C_y rather than by convolution, we are able to produce solutions even though the initial function $u_{01}(0,x,y) = y\theta$ is not defined on all of $\mathbb{R}^2$; and then the solution for $t > 0$, u_{01}, is not globally analytic nor globally smooth. Similarly, for $L = \frac{1}{3}(D_x^3 + D_y^3)$, the Airy function Ai may be used to find $p_t = t^{-2/3}\mathrm{Ai}(-xt^{-1/3})\,\mathrm{Ai}(-yt^{-1/3})$, whereas we can construct local solutions using the vacuums found above and $C_x = x + tD_x^2$, $C_y = y + tD_y^2$. These remarks apply to general H's as well.

Notational Conventions

Before proceeding further we discuss the notation to be used in the following:

(1) Without further comments, x, z, D, y, etc. will denote N-vectors, n, m multi-indices; r, s, λ and other Greek indices will denote single indices.

(2) Summation convention: Repeated Greek indices are assumed summed from 1 to N (e.g., $z \cdot x = z_\lambda x_\lambda$).

(3) R will denote an arbitrary given nonsingular matrix with inverse S. R^* denotes R transpose; similarly for other linear transformations.

(4) For a function $\phi(D)$ we denote $\phi(RD) = \phi(R_{1\lambda} D_\lambda, R_{2\lambda} D_\lambda, \ldots, R_{N\lambda} D_\lambda)$ by $\phi^R(D)$. Similarly $f^R(x) = f(Rx)$.

Linear Transformations, Canonical Variables.

We recall that for $N = 1$ we have

$$\phi(\lambda D) = \lambda^{-xD} \phi \lambda^{xD},$$

$$\lambda^{xD} f(x) = f(\lambda x).$$

We want to determine the effect of a general linear transformation R for $N > 1$.

PROPOSITION 3. *For ϕ a function of D and f a function of x,*

(1) $\phi f^R = (\phi^{R^*} f)^R$; *i.e.*, $\phi(D) f(Rx) = \phi(R^* D) f|_{Rx}$;

(2) $\phi^{R^*} f = (\phi f^R)^S$.

Proof. We check for exponentials $\phi = e^{\alpha \cdot D}$.

(1) $e^{\alpha \cdot D} f(Rx) = f(Rx + R\alpha) = (e^{\alpha \cdot R^* D} f(x))^R = f(x + R\alpha)|_{Rx}$;

(2) $e^{\alpha \cdot R^* D} f(x) = f(x + R\alpha) = (e^{\alpha \cdot D} f(Rx))^S = f(Rx + R\alpha)|_{Sx}$.

Remark. Note that, e.g., in (1), the „R" applies only to the x variable, just as $\lambda^{xD} f(x + \alpha) = f(x\lambda + \alpha)$ for $N = 1$. The above can also be understood in terms of canonical variables. Recall that the usual variables x_j, $z_j (= D_j)$ satisfy the commutation relations $[z_j, x_k] = \delta_{jk}$.

DEFINITION. Any family of operators q_j, p_j, $1 \leqslant j \leqslant N$, satisfying $[p_j, q_k] = \delta_{jk}$ are called canonical variables.

Let us set $q = Rx$, $p = TD$ for some matrix T. Then $[p_j, q_k] = [T_{j\lambda} z_\lambda, R_{k\mu} x_\mu] = T_{j\lambda} R_{k\mu} \delta_{\lambda\mu} = T_{j\lambda} R_{k\lambda} = \delta_{jk}$ implies that $T = S^*$.

We thus have

PROPOSITION 4. *$q_j = R_{j\lambda} x_\lambda$ and $p_j = S_{\lambda j} z_\lambda$ are a family of canonical variables.*

As seen in [3], the associated calculus is the same as that for the usual x, D variables. For example, the relation $\phi(p)f(q) = \phi(D)f(x)|_{x=q}$ is easily seen to be equivalent to Proposition 3.

Action of R on a Generator L

PROPOSITION 5. *Let $L(D)$ have corresponding density $p_t(x)$. Then to $L(RD)$ corresponds the density*

$$p_t^S(x) = |\det S|\, p_t(S^*x).$$

Proof. By Proposition 3 with $R \to R^*$, $e^{tL(RD)}\delta(x-y) = e^{tL(D)}\delta(R^*x - y)|_{S^*x}$. We observe now that for a test function f,

$$\int \delta(R^*x - y)f(y) = f(R^*x) = \int \delta(x - S^*y)f(y)\,|\det S|$$

as follows by the change of variables $y \to R^*y$. That is,

$$\delta(R^*x - y) = |\det S|\,\delta(x - S^*y).$$

We thus conclude that

$$e^{tL(RD)}\delta(x-y) = |\det S|\,p_t(S^*y - x)|_{S^*x} = |\det S|\,p_t(S^*(y-x)).$$

Remark. This proposition relates a linear action in the momentum space to the original configuration space.

Group Actions and Moment Systems

Let $h_n(x, t)$ be a moment system. And let $\mathscr{G}$ be a group acting on $\mathbb{R}^N$. Then, as is well known, the matrices ρ_{mn} satisfying

$$h_m(gx, t) = \sum_n \rho_{mn}(g)\, h_n(x, t)$$

are a representation of $\mathscr{G}$. For example, if $gx = x + y$ we get, for $N = 1$,

$$h_m(x+y, t) = e^{yD}e^{tL}x^m = e^{tL}e^{yD}x^m = e^{tL}(x+y)^m$$

$$= \sum_n \binom{m}{n} y^{m-n} h_n(x, t).$$

Analogous relations hold for $N > 1$.

If L is invariant under (the adjoint action of) a linear group $\mathscr{G}$, the action of $\mathscr{G}$ transfers directly from $x(t)$ to x, i.e., for $R \in \mathscr{G}$, $(Rx(t))^n 1 = h_n(Rx, t)$.

This is seen by applying Proposition 3 as follows (also see Proposition 8 below):

$$e^{z\cdot Rx(t)}1 = e^{tL}e^{z\cdot Rx} = e^{tL(R^*D)}\, e^{z\cdot x}|_{Rx}$$
$$= e^{tL}\, e^{z\cdot x}|_{Rx} = e^{tL(z)+z\cdot Rx}.$$

We can now discuss some further connections with group representations. The discussion will provide some methods useful in the orthogonal theory to follow.

Group Representations and Moment Systems

Recall that the generator of linear actions $x \to \lambda x$ on $\mathbb{R}^1$ is xD. This is proven by setting $\lambda = e^s$ and noting that $u(s, x) = f(e^s x)$ satisfies

$$\frac{\partial u}{\partial s} = x\frac{\partial u}{\partial x}, \qquad u(0, x) = f(x).$$

Now consider $R(s)$ a one-parameter subgroup of a Lie group acting on $\mathbb{R}^N$ with $R(s) = e^{sg}$, g in the Lie algebra. Setting $u(s, x) = f(R(s)\,x)$ we have:

$$\frac{\partial u}{\partial s} = R_{\alpha\mu}(s)\, g_{\mu\lambda} x_\lambda D_\alpha f|_{R(s)x} = g_{\mu\lambda} x_\lambda D_\mu u.$$

That is:

PROPOSITION 6. *The generator of the action $f(x) \to f(R(s)\,x)$ is $gx \cdot D$, where $R(s) = e^{sg}$.*

Remark. The notation $\lambda^{xD} = e^{sxD}$ thus extends in the form $e^{sgx\cdot D}$ which will be denoted more conveniently by R^{xD}, $R = R(s)$. Proposition 3 now may be expressed as:

$$\phi(D)\, R^{xD} = R^{xD}\phi(R^*D).$$

We can now formulate:

THEOREM 1. *Let L be a generator and R a representative of a matrix group. Then there are three classes of representations $\rho_{mn}(R)$:*

(1) *General L,*

$$\rho_{mn}(R) = \frac{1}{n!} D^n R^{\overline{A}}\, x^m|_{x=0}.$$

(2) *L is invariant; i.e., $L(R^*z) = L(z)$,*

$$\rho_{mn}(R) = \frac{1}{n!} D^n\, R^{xD}\, x^m|_{x=0}.$$

(3) *L is a relative invariant; i.e., $L(R^*z)=\chi(R^*)\,L(z)$, χ a character,*

$$\rho_{mn}(R)=\frac{1}{n!}D^n e^{t_S L(D)} R^{xD} x^m\,|_{x=0},$$

where $t_S=t(\chi(S^*)-1)$.

Remark. $R^{\bar{A}}=e^{-tL}R^{xD}\,e^{tL}$ extends $\lambda^{\bar{A}}$. Note that 2 yields a homomorphism $R\to\rho_{mn}(R)$, regardless of L's.

Proof. (1) $h_m(Rx,t)=\sum_n \rho_{mn}(R)\,h_n(x,t)$ can be expressed as $R^{xD}e^{tL}x^m=\sum\rho_{mn}\,e^{tL}x^n$. Applying e^{-tL}, $e^{-tL}R^{xD}e^{tL}x^m=\sum\rho_{mn}\,x^n$. Condition (1) follows.

(2) If L is invariant,

$$R^{\bar{A}}=e^{-tL}R^{xD}e^{tL}=R^{xD}e^{-tL}e^{tL}=R^{xD}.$$

(3) If L is a relative invariant,

$$R^{\bar{A}}=e^{-tL}e^{tL(S^*D)}R^{xD}=e^{t_S L(D)}\qquad\text{as required.}$$

Using this result we can compute the generating function

$$R(a,b)=\sum_{m,n}\frac{a^n b^m}{m!}\rho_{mn}(R).$$

PROPOSITION 7. (1) *For general L,*

$$R(a,b)=\exp(b\cdot Ra+t[L(b)-L(R^*b)]);$$

(2) *For invariant L, $R(a,b)=\exp(b\cdot Ra)$*

(3) *For L a relative invariant,*

$$R(a,b)=\exp(b\cdot Ra+t(1-\chi(R^*))\,L(b)).$$

Proof. Note that (2) and (3) follow easily from (1). For (1),

$$\begin{aligned}R(a,b)&=\sum_{m,n}\frac{a^nD^n}{n!}R^{\bar{A}}\frac{b^m x^m}{m!}\Big|_0=e^{a\cdot D}e^{-tL}R^{xD}e^{tL}e^{b\cdot x}|_0\\&=e^{a\cdot D}R^{xD}e^{t[L(b)-L(R^*b)]}e^{b\cdot x}|_0\\&=e^{t[L(b)-L(R^*b)]}e^{b\cdot Ra}.\end{aligned}$$

Proposition 7 allows us to determine when the action of R is homogeneous in the sense that $h_m(Rx,t)$ splits into a sum of h_n's having the same homogeneity; i.e., $|n|=|m|$. So R acts on the various "floors" indexed by $|m|$.

HOMOGENEITY THEOREM. *L is invariant under R^* if and only if $\rho_{mn}(R) = \rho_{mn}(R)\,\delta_{|m|\,|n|}$.*

Proof. From $R(a,b) = \exp(b \cdot Ra)$, $\sum_n \rho_{mn} a^n = (Ra)^m$. The right-hand side is a product of N sums each term of which, say $(R_{j\lambda} a_\lambda)^{m_j}$, contains m_j factors of a's. Thus each term of $(Ra)^m$ contains $|m|$ factors of a's. In case L is not invariant, for $t \neq 0$, expressions 1 and 3 of Proposition 7 show that homogeneity is not preserved.

We can determine how the C variables transform.

PROPOSITION 8. (1) *For general L,*

$$\rho_{mn}(R) = \frac{1}{n!} D^n (RC_R)^m 1 \,|_{x=0}, \qquad \textit{where } C_R = e^{tL_S} x e^{-tL_S},$$

the C-operator corresponding to the generator $L_S(D) = L(S^*D) - L(D)$.

(2) *For general L,*

$$h_m(Rx, t) = e^{tL(S^*D)}(Rx)^m = e^{tL_S}(RC)^m 1$$

(3) *For invariant L,*

$$h_m(Rx, t) = e^{tL}(Rx)^m = (RC)^m 1;$$

i.e., R factors through.

Proof. (1) $(1/n!)\, D^n R^{\bar{A}} x^m = (1/n!)\, D^n (R^{\bar{A}} x R^{-\bar{A}})^m 1$ since $\bar{A} 1 = 0$. Proceeding,

$$\begin{aligned} R^{\bar{A}} x R^{-\bar{A}} &= e^{-tL} R^{xD} e^{tL} x e^{-tL} S^{xD} e^{tL} \\ &= e^{tL_S} R^{xD} x S^{xD} e^{-tL_S} \\ &= e^{tL_S}(Rx)\, e^{-tL_S} = RC_R. \end{aligned}$$

(2) The first equality follows since $q = Rx$, $p = S^*D$ are a canonical family. Then

$$e^{tL(S^*D)}(Rx)^m = e^{tL_S} e^{tL}(Rx)^m e^{-tL} 1.$$

(3) Follows from 2 since $L_S = 0$ for invariant L.

Further Remarks.

(1) The mapping $g \to gx \cdot D$ is a Lie algebra antihomomorphism; i.e., $[gx \cdot D, g'x \cdot D] = [g', g]\, x \cdot D$.

(2) For $R(s) = e^{sgx \cdot D}$, let $r(s) = \rho_{mn}(R(s))$. Then $r(s)$ is a one-parameter group with generator $g_{mn} = (1/n!)\, D^n\, g\bar{x}(t) \cdot D\, x^m|_0$, using Theorem 1.1.

(3) For L to be invariant, $[gx \cdot D, L] = 0$, i.e., L satisfies $(z \cdot g\nabla) L = 0$, where $\nabla_j = \partial/\partial z_j$. Similarly, L is a relative invariant if and only if L is an eigenfunction of $z \cdot g\nabla$, as follows by differentiating $L(R^*z) = L(e^{sg^*}z) = e^{s\chi}L(z)$. This is a natural generalization of Euler's homogeneity theorem for $z \cdot \nabla$.

(4) Proposition 8 yields hosts of identities as special cases. Let $L = \frac{1}{2}(D_x^2 + D_y^2)$. R orthogonal, e.g., $\left(\begin{smallmatrix}\cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha\end{smallmatrix}\right)$, yields

$$\begin{aligned} H_n(x\cos\alpha + y\sin\alpha, t)\, H_{n'}(-x\sin\alpha + y\cos\alpha, t) &= e^{-tL}(x\cos\alpha + y\sin\alpha)^n(-x\sin\alpha + y\cos\alpha)^{n'} \\ &= ((x - tD_x)\cos\alpha + (y - tD_y)\sin\alpha)^n \\ &\quad \times (-(x - tD_x)\sin\alpha + (y - tD_y)\cos\alpha)^{n'}\, 1 \end{aligned}$$

for Hermite polynomials $H_n(x, t)$.

III. Bernoulli Systems

Definition. A *Bernoulli generator* is a function $L(z)$ such that

(1) $L(z)$ is analytic in some neighborhood of 0 in $\mathbb{C}^N$.

(2) $L(0) = 0$. $(\partial L/\partial z_j)(0) = 0$.

(3) There exist functions ϕ and V, $\phi\colon \mathbb{C}^N \to \mathbb{C}$, $V\colon \mathbb{C}^N \to \mathbb{C}^N$ analytic in a neighborhood of 0, satisfying

(a) V is invertible with analytic inverse;

(b) For any a, b sufficiently near 0, $L(a + b) - L(a) - L(b) = \phi(V_1(a)\, V_1(b), V_2(a)\, V_2(b), \ldots, V_N(a)\, V_N(b))$.

The significance of Bernoulli generators is that they generate systems of orthogonal polynomials that are essentially moment systems. For $N = 1$ we have the following theorem that is basic. It turns out that Meixner had the basic idea in [7] and was led to introduce his classes of polynomials. Our motivation came via probability theory.

Theorem (Meixner-F.) [4]. *The exponential $e^{a\mathcal{C}}1$ has an orthogonal expansion with D mapped into a translation—invariant operator $V(D)$ if and only if L has the form*

$$L(z) = -\frac{\alpha}{\beta}z - \frac{2}{\beta}\log[pe^{Qz/2} + \bar{p}e^{-Qz/2}]$$

or a limiting case ($\alpha = 0$; $Q = 0$; $\beta = 0$; $\alpha = \beta = 0$) *where* α *and* β *are given (complex) numbers.* $Q = \sqrt{\alpha^2 - 2\beta}$, $p = \frac{1}{2} - \alpha/2Q$, $\bar{p} = 1 - p$. *Furthermore,* $V(D) = L'(D)$.

Remarks. (1) In the above, L has been further normalized to $L''(0) = 1$. See [4] for the notation related to Bernoulli generators for $N = 1$.

(2) To an $L(z)$ there is a corresponding density $p_t(x)$. For appropriate choices of α and β, $p_t(x)$ become the transition densities for the basic Markov processes in probability theory, namely, Bernoulli, Poisson, exponential (gamma), and Brownian motion (Gaussian). By using the *local* operator $\bar{C}W$ we can construct, for arbitrary complex α and β, systems of polynomials that generalize the corresponding classical systems which depend on *global* orthogonality relations.

(3) In the following, for $N > 1$, we will continue to suppress vector notations so that, e.g., the basic polynomial $J_n(x, t) \equiv J_{\mathbf{n}}(\mathbf{x}, t) \equiv J_{n_1 n_2 \ldots n_N}(x_1, \ldots, x_N; t)$.

(4) The operator $V(D)$ acts as the lowering or differentiation (gradient) operator. $V_j(D) J_n(x, t) = n_j J_{n-e_j}(x, t)$; recalling the notation e_j for the standard basis on $\mathbb{Z}^N$, $e_{jk} = \delta_{jk}$.

(5) The following standard notations will be adopted:

(a) l_j will denote a standard one-dimensional Bernoulli generator with parameters α_j and β_j determined by the characteristic equation $l_j'' = 1 + \alpha_j l_j' + \frac{1}{2}\beta_j l_j'^2$.

(b) c will denote a diagonal matrix with entries $c_{jj} \equiv c_j$, $1 \leqslant j \leqslant N$.

(c) $\mathscr{L}(z) \equiv \sum_{j=1}^N c_j l_j(z_j)$ is a linear combination of one-dimensional generators.

(d) $\varepsilon_{jk} = (\partial^2 L/\partial z_j \partial z_k)(0)$ is a general nonsingular symmetric matrix; i.e., L is not assumed to be normalized; e.g., ε_{jk} is not necessarily diagonal.

(e) L_j, L_{jk}, L_{jkl}, etc. will denote the various partial derivatives of $L(z)$, i.e., $L_j = \partial L/\partial z_j$, $L_{jk} = \partial^2 L/\partial z_j \partial z_k$, and so on. $L' = (L_j)$, $L'' = (L_{jk})$, etc.

LEMMA. *A one-dimensional generator* L, *with* $L(0) = L'(0) = 0$, *satisfying* $L'' = c + aL' + bL'^2$ *is of the form* $L = cl$, *where* l *is a Bernoulli generator with parameters* $\alpha = a$, $\beta = 2cb$.

Proof. From $l'' = 1 + \alpha l' + \frac{1}{2}\beta l'^2$ we have $(cl)'' = c + \alpha(cl)' + \frac{1}{2}\beta c^{-1}(cl')^2$; i.e., $L'' = c + \alpha L' + \frac{1}{2}\beta c^{-1} L'^2$.

DEFINITION. A Bernoulli generator is *separable* if it is of the general form $\sum_{j=1}^N f_j(R_{j\lambda} z_\lambda)$ for some nonsingular R.

THEOREM 2. (1) *A (nondegenerate) general Bernoulli generator is determined by a system of characteristic equations of the form*

$$L_{jk} = \varepsilon_{jk} + a^{\lambda}_{jk} L_{\lambda} + B^{\lambda\mu}_{jk} L_{\lambda} L_{\mu}.$$

(2) *A separable Bernoulli generator is of the form* $\mathscr{L}^R = \sum_{j=1}^{N} c_j l_j(R_{j\lambda} z_{\lambda})$. *For parameters* α_j, β_j *corresponding to* l_j:

(a) $\varepsilon = R^* c R$, *i.e.*, $\varepsilon_{jk} = c_{\lambda} R_{\lambda j} R_{\lambda k}$;

(b) $a^{l}_{jk} = \alpha_{\lambda} R_{\lambda j} R_{\lambda k} S_{l\lambda}$;

(c) $B^{lm}_{jk} = \frac{1}{2}\beta_{\lambda} c_{\lambda}^{-1} R_{\lambda j} R_{\lambda k} S_{l\lambda} S_{m\lambda}$.

Remarks. (1) "Nondegenerate" means essentially that the components of the corresponding process are not dependent. Technically it is the assumption of the nonsingularity of ε and of the $\mathscr{O}$ matrix below.

(2) The separable generators are a particularly interesting tractable class. The "general solution" is not at all apparent.

Proof.

Step 1. Expand the relation $L(a+b) - L(a) - L(b) = \phi(V(a)\, V(b))$ around $a = 0$:

$$0 = \phi(V_1(0)\, V_1(b), \dots, V_N(0)\, V_N(b)),$$

thus $V(0) = 0$.

$$L_j(b) - L_j(0) = V_{\lambda}(b)\, V_{\lambda j}(0)\, \phi_{\lambda}(0),$$

thus $L_j(b) = V_{\lambda j}(0)\, \phi_{\lambda}(0)\, V_{\lambda}(b) \equiv \mathscr{O}_{j\lambda} V_{\lambda}(b)$.

$$\begin{aligned} L_{jk}(b) - \varepsilon_{jk} &= V_{\lambda}(b)\, V_{\lambda jk}(0)\, \phi_{\lambda}(0) \\ &\quad + V_{\lambda}(b)\, V_{\mu}(b)\, V_{\lambda j}(0)\, V_{\mu k}(0)\, \phi_{\lambda\mu}(0) \\ &= A^{\lambda}_{jk} V_{\lambda}(b) + b^{\lambda\mu}_{jk} V_{\lambda}(b)\, V_{\mu}(b) \end{aligned}$$

Step 2. Setting $o = \mathscr{O}^{-1}$, $V = oL'$ and the characteristic equations result:

$$L_{jk} = \varepsilon_{jk} + a^{\lambda}_{jk} L_{\lambda} + B^{\lambda\mu}_{jk} L_{\lambda} L_{\mu}.$$

Furthermore,

$$L_{jkl}(0) = a^{\lambda}_{jk} \varepsilon_{\lambda l};$$

$$L_{jklm}(0) = a^{\lambda}_{jk} a^{\mu}_{\lambda l} \varepsilon_{\mu m} + B^{\lambda\mu}_{jk} \gamma_{\lambda\mu lm}, \qquad \gamma_{pqlm} = \varepsilon_{pl}\varepsilon_{qm} + \varepsilon_{ql}\varepsilon_{pm}.$$

Step 3. For separable solutions substitute $z \to Rz$, $L(z) = \sum_j \psi_j(R_{j\lambda} z_\lambda)$. Then

$$R_{\lambda j} R_{\lambda k} \psi''_\lambda = \varepsilon_{jk} + a^\lambda_{jk} R_{\mu\lambda} \psi'_\mu + B^{\lambda\mu}_{jk} R_{\sigma\lambda} R_{\zeta\mu} \psi'_\sigma \psi'_\zeta$$

The cross terms $\psi'_\sigma \psi'_\zeta$ may be eliminated since there are $N(N+1)/2$ equations and only $N(N-1)/2$ cross terms. Thus there is an equation relating ψ's of independent arguments yielding N equations of type $\psi''_j = k_j + a_j \psi'_j + b_j \psi'^2_j$. By the Lemma, there follows the representation $L(z) = \sum_j c_j l_j(R_{j\lambda} z_\lambda)$.

Step 4. Now beginning with the representation of $L(z)$ as $\mathscr{L}^R$, substitution for $L_j(z) = c_\lambda l'_\lambda(R_{\lambda\mu} z_\mu) R_{\lambda j}$, and similarly for L_{jk}, L_{jkl} yields (2a), (2b), using $l''_j{}'(0) = \alpha_j$. For (2c), substitute for L_{jklm} then use (2a), (2b) and $l^{iv}_j(0) = \alpha^2_j + \beta_j$. This yields $B^{\lambda\mu}_{jk} \gamma_{\lambda\mu lm} = c_\lambda \beta_\lambda R_{\lambda j} R_{\lambda k} R_{\lambda l} R_{\lambda m}$. It is readily checked that this agrees with (2c). (See the next proposition for a derivation of (2c).)

The tensor character of a^l_{jk} and B^{lm}_{jk} can be described as follows. The gradient of L, $L' = (L_j)$, is a covariant vector under the transformation $z \to Rz$. Since $L_j \to R_{\lambda j} L_\lambda$ and $L_{jk} \to R_{\lambda j} R_{\mu k} L_{\lambda\mu}$, L_{jk} is a covariant 2-tensor. A contravariant vector L^j transforms as $L^j \to S_{j\lambda} L^\lambda$ so that $L^\lambda L_\lambda$ is invariant.

PROPOSITION 9. *Under the linear transformation $z \to Rz$, a^l_{jk} is a 3-tensor covariant in j, k and contravariant in l, B^{lm}_{jk} is a 4-tensor covariant in j, k, contravariant in l, m.*

Proof. From the equation $L_{jk} = \varepsilon_{jk} + a^\lambda_{jk} L_\lambda + B^{\lambda\mu}_{jk} L_\lambda L_\mu$ we have:

(1) $\varepsilon_{jk} = L_{jk}(0)$ is covariant in j, k.

(2) The covariance of L_j implies the contravariance properties of a and B.

(3) The covariance in jk of a and B can be explicitly checked from L_{jkl}, L_{jklm}. For a^l_{jk} it follows readily from $L_{jkl}(0)$.
From $L_{jklm}(0) = a^\lambda_{jk} a^\mu_{\lambda l} \varepsilon_{\mu m} + B^{\lambda\mu}_{jk} \gamma_{\lambda\mu lm}$, we can derive, denoting by $\bar{B}^{lm}_{jk}$ the transformed B,

$$B^{\lambda\mu}_{jk} \gamma_{\lambda\mu lm} = \bar{B}^{\delta\epsilon}_{\alpha\beta} S_{\alpha j} S_{\beta k} R_{\sigma\delta} R_{\zeta\epsilon} \gamma_{\sigma\zeta lm}$$

using the covariance of γ in its indices. It must be checked that $X^{\lambda\mu} \gamma_{\lambda\mu lm} = 0$ implies $X = 0$. In any case, we can take $\varepsilon_{jk} = \varepsilon_\lambda R_{\lambda j} R_{\lambda k}$ with $\varepsilon_j \neq 0$, $1 \leqslant j \leqslant N$. Then it follows without difficulty that X must be skew-symmetric. B can be assumed symmetric in lm as B^{lm}_{jk} are coefficients of the quadratic form $B^{\lambda\mu}_{jk} L_\lambda L_\mu$ appearing in the characteristic equations. So we take X to be symmetric and hence conclude $X = 0$. That is, the symmetry of B^{lm}_{jk} allows cancellation of the γ's. The required tensor properties of B_{jk}^{lm} then readily follow.

Remark. Proposition 9 and the Lemma, using $a_j \to \alpha_j$, $b_j \to \frac{1}{2}\beta_j c_j^{-1}$ now give us 2*b* and 2*c* of Theorem 2. The special "diagonal tensors" α_j and $\frac{1}{2}\beta_j c_j^{-1}$ of the separable case are treated simply as contractions $\alpha_j = a_{\lambda\mu}^{\sigma} \delta_j^{\lambda} \delta_j^{\mu} \delta_{\sigma j}$, $\frac{1}{2}\beta_j c_j^{-1} = B_{\lambda\mu}^{\sigma\tau} \delta_j^{\lambda} \delta_j^{\mu} \delta_{\sigma j} \delta_{\tau j}$. For the properties of a and B under general transformations $z \to \psi(z)$, see the last section. Presently we proceed with describing the canonical structure associated with a separable Bernoulli generator.

THEOREM 3. *Corresponding to a (nondegenerate) separable Bernoulli generator $L(z)$ are the following:*

(1) *The canonical operator $V(D) = c^{-1} S^* L' = v^R(D)$, where $v_j(z_j)$ is the canonical operator associated with l_j; i.e., $v_j = l_j'$.*

(2) *The canonical variable $\xi_j = S_{\lambda j} x_\lambda w_j(R_{j\mu} z_\mu)$.*

(3) *$J_n(x, t) = \prod_{j=1}^{N} J_{n_j}(S_{\lambda j} x_\lambda, c_j t)$ are the orthogonal polynomials, where J_{n_j} corresponds to l_j.*

(4) *The density $p_t(x) = |\det S| \prod_{j=1}^{N} p_{c_j t}^{(j)}(S_{\lambda j} x_\lambda)$.*

Proof. Corresponding to $L(z) = \sum l_j(z_j)$ is the density $\prod_{j=1}^{N} p_t^{(j)}(x_j)$. By Proposition 5, (4) follows for the general $\mathscr{S}^R$. From this and Proposition 4 it is seen that $x \to S^* x$ corresponds to the action $z \to Rz$. Thus (2) and (3) follow. Set $J_n(x, t) = \mathscr{T}_n(S^* x, t)$, $\mathscr{T}_n(x, t) = \prod_j J_{n_j}(x_j, c_j t)$. Then $R^{*xD} J_n(x, t) = \mathscr{T}_n(x, t)$ and so $V = S^{*xD} v R^{*xD} = v(RD)$ by Proposition 3. Finally, $L' = R_{\lambda j} c_\lambda l_\lambda'(R_{\lambda\mu} z_\mu) = R^* c\, V$, so $V = c^{-1} S^* L'$.

Remarks.

1. Thus the c_j are *characteristic speeds* or time-scaling factors.

2. Recall that in the proof of Theorem 2 there is a free matrix $\mathscr{O}$ such that $L' = \mathscr{O} V$. In the canonical separable case, by Theorem 3.1, $\mathscr{O} = R^* c$. However, we still have to prove a uniqueness theorem ruling out other $\mathscr{O}$'s. The choice above we call the *canonical choice*. Next we will study the action $V \to \mathscr{O} V$ in detail.

Orthogonal Mappings and Uniqueness of the Diagonal Representation

For $N = 1$, the polynomials J_n are naturally ordered by degree and uniquely determined by normalization. In the independent ($R = I$) separable case, for $N > 1$, the J_n are products of 1-dimensional J's and so we have the variance $j_n = \langle J_n^2 \rangle = \prod_k j_{n_k}(c_k t)$, $j_{n_k} = \langle J_{n_k}^2 \rangle$. However, in general only the order $|n|$ provides a natural analog of degree. This will be clarified by studying the action $V \to \mathscr{O} V$.

DEFINITION. A matrix $\mathcal{O}$ is *semi-orthogonal* if there are diagonal matrices k_1 and k_2 such that $\mathcal{O}^* k_1 \mathcal{O} = k_2$.

Fix L. Assume that V and J_n are the canonical choices and corresponding to $\mathcal{O}V$ is the orthogonal sequence K_n such that $(\mathcal{O}_{j\lambda} V_\lambda) K_n = n_j K_{n-e_j}$. Define P by $K_n = \sum_m P_{nm} J_m$. K_0 is taken to be 1 (this follows anyway from $L(0) = V(0) = 0$). The orthogonality of K_n yields $\langle K_n K_m \rangle \equiv k_n \delta_{nm} = \sum_l P_{nl} P_{ml} j_l$; i.e., P is semi-orthogonal. Furthermore $K_0 = 1 = J_0$ implies $P_{0m} = \delta_{0m}$ and also, e.g.,

$$\delta_{0n} = \langle K_0 K_n \rangle = \sum_l P_{nl} \delta_{0l} j_l = P_{n0}.$$

As seen in Section II, the map $\mathcal{O} \to P_{nm}(\mathcal{O})$ is a representation. By the basic expansion theorem (see e.g., [4 p. 26, 27, 92]), as in Theorem 1,

$$P_{nm} = \frac{1}{m!} e^{tL} V^m K_n \big|_{x=0}.$$

Thus, with $U = V^{-1}$, $M = L^\circ U$, $(\mathcal{O}V)^{-1} = U_o$, so

$$\begin{aligned}\sum \frac{a^n}{n!} b^m P_{nm} &= e^{tL}\, e^{bV}\, e^{x \cdot U(o a) - tM(o a)} \big|_{x=0} \\ &= e^{b \cdot o a}.\end{aligned}$$

Note that $o_{jk} = P_{e_k e_j}$, so that o can be recovered from P. As in the Homogeneity Theorem of Section II, we conclude that K_n has the same order as J_n, i.e., $P_{nm} = \delta_{|n|\,|m|} P_{nm}$. Equating coefficients of b^m yields

$$\sum \frac{a^n}{n!} P_{nm} = \frac{1}{m!} (o a)^m, \qquad P_{nm} = \frac{1}{m!} D^n (o x)^m \bigg|_{x=0}.$$

The semi-orthogonality of P now implies

$$\sum \frac{a^n b^n}{n!^2} k_n = \sum \frac{(o a)^n (o b)^n}{n!^2} j_n.$$

We can now prove

THEOREM 4. *For a separable Bernoulli system the canonical choices are unique except when L is a mixture of Poisson or Gaussian generators in which case V is determined up to a semi-orthogonal transformation.*

Proof. For $N = 1$, $\beta_j \to 2\beta_j$, $t \to c_j t \equiv t_j$,

$$(n_j!)^{-1} j_{n_j} = t_j (t_j + \beta_j) \cdots (t_j + (n_j - 1)\beta_j) = (t_j e^{\beta_j T_j})^{n_j} 1$$

where $T_j = d/dt_j$. Set $\tau_j = t_j e^{\beta_j T_j}$. The t_j are treated as independent variables. Then

$$\sum \frac{(\wp a)^n(\wp b)^n}{n!^2} j_n = e^{\wp_{\mu\lambda} a_\lambda \wp_{\mu\epsilon} b_\epsilon \tau_\mu} 1 = \sum \frac{a^n b^n}{n!^2} k_n \equiv k(ab).$$

Since $k(ab)$ is invariant with respect to scalings $a_j \to y a_j$, $b_j \to y^{-1} b_j$, it follows that $\wp_{\mu\lambda} \wp_{\mu\epsilon} \tau_\mu = \delta_{\lambda\epsilon} \sigma_\lambda$; this serving to define σ. Then

$$k(ab) = e^{a_\lambda \sigma_\lambda b_\lambda} 1 \qquad \text{and} \qquad (n!)^{-1} k_n = \sigma^n 1.$$

The critical relations are those that give zero. For example, for $j \neq k$

$$\wp_{\mu j} \wp_{\mu k} t_\mu e^{\beta_\mu T_\mu} = 0.$$

Applying this to a function $f(t_1, \ldots, t_N)$

$$\wp_{\mu j} \wp_{\mu k} t_\mu f(\mathbf{t} + \beta_\mu e_\mu) = 0 \qquad \text{for } all \text{ functions } f.$$

Thus unless $\boldsymbol{\beta} = 0$, $\wp$ must be diagonal and so only trivial scalings of the v_j are induced.

In the case $\boldsymbol{\beta} = 0$, i.e., Gaussian or Poisson processes,

$$\wp_{\mu j} \wp_{\mu k} t_\mu = \delta_{jk} \sigma_j$$

obtains if and only if $\wp^* c\, \wp$ is diagonal; i.e., $\wp$ is semi-orthogonal.

This completes the discussion of the separable structure. Next we discuss various group actions.

Scaling Transformations

PROPOSITION 10. *For $N = 1$, $k > 0$, the parabolic scalings $x \to kx$, $t \to k^2 t$, $\alpha \to k\alpha$, $\beta \to k^2 \beta$ transform $\bar{C}W \to k\bar{C}W$, $V \to k^{-1}V$, $J_n(x, t) \to k^n J_n(x, t)$ and $\bar{C}WV$ is invariant.*

Proof. For $x \to kx$, $z \to k^{-1} z$. $Q = \sqrt{\alpha^2 - 2\beta} \to kQ$ so that Qz is an invariant. From $V = 2qF(Qz)$, $W = G(Qz)$ (for the explicit forms of F and G see [4 p. 52, 1]) follow $V \to k^{-1} V$ and the invariance of W. Thus $\bar{C}W = xW - tVW \to kxW - k^2 t\, k^{-1} VW = k\bar{C}W$. The results for $J_n = (\bar{C}W)^n 1$ and $\bar{C}WV$ now follow.

We can extend these scaling transformations to $N > 1$.

PROPOSITION 11. *Let $L = \mathscr{L}^R$ be a separable Bernoulli generator. Let k denote an arbitrary positive definite diagonal matrix with diagonal entries k_j. Then the scaling $x \to R^* k S^* x$, $c_j \to k_j^2 c_j$, $\alpha_j \to \alpha_j k_j$, $\beta_j \to \beta_j k_j^2$ induces $J_n(x, t) \to (\prod_j k_j^{n_j}) J_n(x, t)$.*

Proof. $J_n(x,t)=\prod_j J_{n_j}(S_{\lambda j}x_\lambda, c_j t)$. $S^*x \to kS^*x$ and so Proposition 10 applies.

Matrix Transformations

The discussion centered around Theorem 1 carries over to the Bernoulli systems. Thus, for a linear transformation E (so as to avoid confusion with R above):

PROPOSITION 12. *Let $\sigma_{mn}(E)$ be defined by $J_m(Ex,t)=\sum_n \sigma_{mn}(E)\,J_n(x,t)$. Then:*

(1) (a) *For general L, $\sigma_{mn}(E)=(1/n!)\,V^n\,E^A\,\xi^m\,1|_{x=0}$;*

(b) *Invariant L yields $\sigma_{mn}(E)=(1/n!)\,V^n\,E^{xD}\xi^m\,1|_{x=0}$.*

2. *The generating functions are:*

(a) *General L,*

$$\sum\sum \frac{a^n b^m}{m!}\sigma_{mn}(E)=\exp[a\cdot V(E^*U(b))+t(L(E^*U(b))-M(b))]$$

(b) *Invariant L,*

$$\sum\sum \frac{a^n b^m}{m!}\sigma_{mn}(E)=\exp[a\cdot V(E^*U(b))].$$

Proof. (1) (a) $J_m(Ex,t)=E^{xD}e^{-tL}\xi^m 1$. So

$$\sigma_{mn}=\frac{1}{n!}e^{tL}V^n E^{xD}e^{-tL}\xi^m 1\bigg|_{x=0},$$

using the basic expansion theorem (recall the discussion preceding Theorem 4).

(b) Follows immediately from (a).

2. For (a), using $(1(a))$,

$$\begin{aligned}\sum\sum \frac{a^n b^m}{m!}\sigma_{mn}(E)&=e^{tL}e^{a\cdot V}E^{xD}e^{-tL}e^{b\xi}1|_{x=0}\\&=e^{a\cdot V}e^{tL}e^{Ex\cdot U(b)-tM(b)}|_0\\&=e^{a\cdot V}e^{tL}e^{x\cdot E^*U(b)-tM(b)}|_0\end{aligned}$$

which yields (2(a)). Relation (2(b)) then follows.

Remarks. The case of L relatively invariant is easily deduced from the general case.

Tensor Formulation of the Characteristic Equations

In the above we studied the behavior of the characteristic equations under linear transformations $z \to Rz$. We conclude by considering a general smooth transformation $z = \phi(\bar{z})$. Transformed quantities in general will be denoted by bars. For convenience set $\phi_{jk} = \partial\phi_j/\partial\bar{z}_k$, consistent with the linear case, and $\phi_{jkl} = \partial^2\phi_j/\partial\bar{z}_k\partial\bar{z}_l$.

Set $\bar{L}(\bar{z}) = L(\phi(\bar{z}))$. Then $\bar{L}_j = \partial\bar{L}/\partial\bar{z}_j = L_\alpha\phi_{\alpha j}$; i.e., L_j is a covariant tensor. The characteristic equations for L are

$$\frac{\partial L_j}{\partial z_k} = \varepsilon_{jk} + a_{jk}^{\lambda}L_\lambda + B_{jk}^{\lambda\mu}L_\lambda L_\mu.$$

We can allow a and B to be "general" functions, rather than necessarily constants, symmetric in the indices jk, B symmetric in its upper indices. Now define the covariant derivative $L_{j,k} = \partial L_j/\partial z_k - a_{jk}^{\lambda}L_\lambda$, i.e., the a_{jk}^{l} are now to be "Christoffel symbols." It must be checked that everything works correctly.

THEOREM 5. *The equation*

$$L_{j,k} = \varepsilon_{jk} + B_{jk}^{\lambda\mu}L_\lambda L_\mu$$

is a tensor equation. ε_{jk} *is a symmetric covariant tensor; and* B_{jk}^{lm} *is covariant in* j, k, *contravariant in* l, m.

Proof. L_{jk} will still denote $\partial L_j/\partial z_k$. Then

$$\bar{L}_{jk} = \partial\bar{L}_j/\partial\bar{z}_k = L_{\alpha\beta}\phi_{\alpha j}\phi_{\beta k} + L_\alpha\phi_{\alpha jk}.$$

For $\bar{L}$ to be a Bernoulli generator,

$$\bar{L}_{jk} = \bar{\varepsilon}_{jk} + \bar{a}_{jk}^{\lambda}\bar{L}_\lambda + \bar{B}_{jk}^{\lambda\mu}\bar{L}_\lambda\bar{L}_\mu$$

or

$$L_{\alpha\beta}\phi_{\alpha j}\phi_{\beta k} + L_\alpha\phi_{\alpha jk} = \bar{\varepsilon}_{jk} + \bar{a}_{jk}^{\lambda}\phi_{\mu\lambda}L_\mu + \bar{B}_{jk}^{\lambda\mu}L_\delta L_\epsilon\phi_{\delta\lambda}\phi_{\epsilon\mu}$$

Thus, substituting for $L_{\alpha\beta}$ and comparing,

$$\bar{\varepsilon}_{jk} = \varepsilon_{\alpha\beta}\phi_{\alpha j}\phi_{\beta k},$$

$$\bar{B}_{jk}^{\lambda\mu}\phi_{l\lambda}\phi_{m\mu} = B_{\alpha\beta}^{lm}\phi_{\alpha j}\phi_{\beta k} \qquad \text{appropriately,}$$

while

$$\bar{a}^{\lambda}_{jk}\phi_{l\lambda} = \phi_{ljk} + a^{l}_{\alpha\beta}\phi_{\alpha j}\phi_{\beta k}$$

so that $(L_{\alpha\beta} - a^{\lambda}_{\alpha\beta}L_{\lambda})\,\phi_{\alpha j}\phi_{\beta k} = \bar{L}_{jk} - \bar{a}^{\lambda}_{jk}\bar{L}_{\lambda}$, as required.

The "Euclidean space" $a = 0$ corresponds to symmetric processes. With $B = 0$, the equations $L_{j,k} = \varepsilon_{jk}$ correspond to Poisson processes for $a \neq 0$ and Gaussian processes for $a = 0$.

As may be expected, if we can consider the a^{l}_{jk} as "connection coefficients," then the B's determine the associated "curvature tensor."

DEFINITION. Denote the lowering of an upper index by contraction with ε by a vertical bar; e.g., $a^{\lambda}_{jk}\varepsilon_{\lambda l} = a_{jk|l}$, $B^{\lambda m}_{jk}\varepsilon_{\lambda l} = B^{m}_{jk|l}$. Denoting by A^{m}_{jkl} the curvature tensor corresponding to the a^{l}_{jk}, for a^{l}_{jk} independent of z,

$$A^{m}_{jkl} = a^{\lambda}_{jl}a^{m}_{\lambda k} - a^{\lambda}_{jk}a^{m}_{\lambda l}.$$

Set $b^{mno}_{jkl} = B^{\lambda m}_{jk}B^{no}_{\lambda l}$. Notice that

$$B^{lm}_{jk,r} = B^{\lambda m}_{jk}a^{l}_{\lambda r} + B^{l\mu}_{jk}a^{m}_{\mu r} - B^{lm}_{\alpha k}a^{\alpha}_{jr} - B^{lm}_{j\beta}a^{\beta}_{kr}.$$

Anti-symmetrization is denoted by brackets; e.g., $F_{[kl]} = F_{kl} - F_{lk}$. Symmetrization is denoted by parentheses.

PROPOSITION 13. *The following relations hold:*

(1) $\frac{1}{2}A^{m}_{jkl} = B^{m}_{j[k|l]}$

(2) $a_{jk|l}$ *is symmetric in jkl.*

(3) $b^{(mno)}_{jkl}$ *is symmetric in jkl.*

(4) $B^{lm}_{j[k,r]} = 0.$

Proof.

$$\begin{aligned} L_{jkl} &= a^{\lambda}_{jk}L_{\lambda l} + 2B^{\lambda\mu}_{jk}L_{\lambda l}L_{\mu} \\ &= a_{jk|l} + (2B^{\mu}_{jk|l} + a^{\lambda}_{jk}a^{\mu}_{\lambda l})\,L_{\mu} \\ &\quad + (2B^{\lambda\beta}_{jk}a^{\alpha}_{\lambda l} + a^{\lambda}_{jk}B^{\alpha\beta}_{\lambda l})L_{\alpha}L_{\beta} + 2B^{\lambda\mu}_{jk}B^{\alpha\beta}_{\lambda l}L_{\alpha}L_{\beta}L_{\mu}. \end{aligned}$$

Since $L(0) = L'(0) = 0$, $L''(0) = \varepsilon$, L has leading term $\frac{1}{2}\varepsilon_{\lambda\mu}z_{\lambda}z_{\mu}$ and L_j has leading term $\varepsilon_{j\mu}z_{\mu}$. Thus L_{μ}, $L_{\alpha}L_{\beta}$ and $L_{\alpha}L_{\beta}L_{\mu}$ are independent. From symmetry of L_{jkl}, the zero order terms yield (2), the L_{μ} terms yield (1), the $L_{\alpha}L_{\beta}$ terms yield (4) and the third-order terms imply (3).

Remark. The above relations hold for a and B functions of z as well, with A^{m}_{jkl} indeed being the curvature tensor associated with $a^{l}_{jk}(z)$.

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